

ON QUANTUM ANALOGUES OF P-BRANE BLACK HOLES

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In a multidimensional model with several scalar fields and an m -form we deal with classical spherically symmetric solutions with one (electric or magnetic) p -brane and Ricci-flat internal spaces and the corresponding solutions to the Wheeler–DeWitt (WDW) equation. Classical black holes are considered and their quantum analogues (e.g. for $M2$ and $M5$ extremal solutions in $D = 11$ supergravity, electric and magnetic charges in $D = 4$ gravity) are suggested when the curvature coupling in the WDW equation is zero.

1. Introduction

In this paper we continue our investigations of p -brane solutions (see, e.g., [1, 2, 3] and references therein) based on the sigma-model approach [4, 5, 6].

The model under consideration contains several scalar dilatonic fields and one antisymmetric form. We consider spherically symmetric solutions (see [15, 14]), when all functions depend on one radial variable and pay attention to black hole (BH) solutions with Ricci-flat internal spaces (see [15, 14] and special solutions in [16, 17, 18]).

The corresponding solutions to the WDW equation (in the spherically symmetric case) were considered in [12, 14]. Here we use the covariant “d’Alembertian” (and/or conformally-covariant) form of the WDW equation of Refs. [9, 10, 11]. We single out certain classes of solutions to the WDW equations and suggest quantum analogues of BH solutions.

The plan of the paper is as follows. In Sec. 2 we consider the model and present the WDW equation. Sec. 3 is devoted to classical and quantum exact solutions when spherical symmetry is assumed. In Sec. 4 we consider the classical black-hole solutions and for the case $a = 0$ (the term $aR[\mathcal{G}]$ in the WDW equation is responsible for the scalar curvature of minisupermetric \mathcal{G}) we suggest quantum analogues of black-hole solutions. In the extremal case we consider several examples, e.g. for $D = 4$ Einstein-Maxwell theory and $D = 11$ supergravity. It is shown that the brane part of the solution satisfying the outgoing-wave boundary condition is regular for small brane quasi-volume (it looks like a pseudo-Euclidean quantum wormhole). We also compare our approach with that suggested by H. Lü, J. Maharana, S. Mukherji and C.N. Pope [20] (with flat minisupermetric and classical fields of forms).

2. The model

Consider the model governed by the action

$$S = \frac{1}{2\kappa^2} \int_M d^D z \sqrt{|g|} \{ R[g] - h_{\alpha\beta} g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta - \frac{1}{m!} \exp[2\lambda(\varphi)] F^2 \} + S_{GH} \quad (2.1)$$

where $g = g_{MN} dz^M \otimes dz^N$ is the metric ($M, N = 1, \dots, D$), $\varphi = (\varphi^\alpha) \in \mathbb{R}^l$ is a vector of dilatonic scalar fields, $(h_{\alpha\beta})$ is a non-degenerate $l \times l$ matrix ($l \in \mathbb{N}$), $F = dA = \frac{1}{m!} F_{M_1 \dots M_m} dz^{M_1} \wedge \dots \wedge dz^{M_m}$ is an m -form ($m \geq 1$) on a D -dimensional manifold M and λ is a 1-form on \mathbb{R}^l : $\lambda(\varphi) = \lambda_\alpha \varphi^\alpha$, $\alpha = 1, \dots, l$. In (2.1) we denote $|g| = |\det(g_{MN})|$, $F^2 = F_{M_1 \dots M_m} F_{N_1 \dots N_m} g^{M_1 N_1} \dots g^{M_m N_m}$, where S_{GH} is the standard Gibbons-Hawking boundary term [8]. The signature of the metric is $(-1, +1, \dots, +1)$.

The equations of motion corresponding to (2.1) have the following form:

$$R_{MN} - \frac{1}{2} g_{MN} R = T_{MN}, \quad (2.2)$$

$$\Delta[g] \varphi^\alpha - \sum_{a \in \Delta} \frac{\lambda^a}{m!} e^{2\lambda(\varphi)} F^2 = 0, \quad (2.3)$$

$$\nabla_{M_1}[g](e^{2\lambda(\varphi)} F^{M_1 \dots M_m}) = 0, \quad (2.4)$$

$\alpha = 1, \dots, l$. In (2.3) $\lambda^\alpha = h^{\alpha\beta} \lambda_\beta$, where $(h^{\alpha\beta})$ is matrix inverse to $(h_{\alpha\beta})$. In (2.2)

$$T_{MN} = T_{MN}[\varphi, g] + e^{2\lambda(\varphi)} T_{MN}[F, g], \quad (2.5)$$

where

$$T_{MN}[\varphi, g] = h_{\alpha\beta} \left(\partial_M \varphi^\alpha \partial_N \varphi^\beta - \frac{g_{MN}}{2} \partial_P \varphi^\alpha \partial^P \varphi^\beta \right),$$

$$T_{MN}[F, g] = \frac{1}{m!} \left[-\frac{1}{2} g_{MN} F^2 + m F_{M M_2 \dots M_m}^a F_N^{M_2 \dots M_m} \right].$$

In (2.3), (2.4) $\Delta[g]$ and $\nabla[g]$ are the Laplace-Beltrami and covariant derivative operators corresponding to g , respectively.

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Consider the manifold

$$M = \mathbb{R} \times (M_0 = S^{d_0}) \times (M_1 = \mathbb{R}) \times \dots \times M_n \quad (2.6)$$

with the metric

$$g = e^{2\gamma(u)} du \otimes du + \sum_{i=0}^n e^{2\phi^i(u)} g^i, \quad (2.7)$$

where u is a radial coordinate, $g^i = g_{m_i n_i}^i(y_i) dy_i^{m_i} \otimes dy_i^{n_i}$ is a metric on M_i satisfying the equation

$$R_{m_i n_i}[g^i] = \xi_i g_{m_i n_i}^i, \quad (2.8)$$

$m_i, n_i = 1, \dots, d_i$; $d_i = \dim M_i$, $\xi_i = \text{const}$, $i = 0, \dots, n$; $n \in \mathbb{N}$. Thus (M_i, g^i) are Einstein spaces. The functions $\gamma, \phi^i: (u_-, u_+) \rightarrow \mathbb{R}$ are smooth. The metric g^0 is a canonical metric on $M_0 = S^{d_0}$, $\xi_0 = d_0 - 1$ and $g^1 = -dt \times dt$, $\xi_1 = 0$. Here (M_1, g^1) is a time manifold.

Each manifold M_i is assumed to be oriented and connected, $i = 0, \dots, n$ (for $i = 0, 1$ this is satisfied automatically). Then the volume d_i -form

$$\tau_i = \sqrt{|g^i(y_i)|} dy_i^1 \wedge \dots \wedge dy_i^{d_i}, \quad (2.9)$$

and the signature parameter

$$\varepsilon(i) = \text{sign det}(g_{m_i n_i}^i) = \pm 1 \quad (2.10)$$

are correctly defined for all $i = 0, \dots, n$. Here $\varepsilon(0) = 1$ and $\varepsilon(1) = -1$.

Let Ω_0 be a set of all subsets of $I_0 \equiv \{0, \dots, n\}$: $\Omega_0 = \{\emptyset, \{0\}, \{1\}, \dots, \{n\}, \{0, 1\}, \dots, \{0, 1, \dots, n\}\}$. For any $I = \{i_1, \dots, i_k\} \in \Omega_0$, $i_1 < \dots < i_k$, we define the form

$$\tau(I) \equiv \tau_{i_1} \wedge \dots \wedge \tau_{i_k}, \quad (2.11)$$

of rank

$$d(I) \equiv \sum_{i \in I} d_i, \quad (2.12)$$

and the corresponding p -brane submanifold

$$M_I \equiv M_{i_1} \times \dots \times M_{i_k}, \quad (2.13)$$

where $p = d(I) - 1$, $\dim M_I = d(I)$. We also define the ε -symbol

$$\varepsilon(I) \equiv \varepsilon(i_1) \dots \varepsilon(i_k). \quad (2.14)$$

For $I = \emptyset$ we put $\tau(\emptyset) = \varepsilon(\emptyset) = 1$, $d(\emptyset) = 0$.

For the field of form we adopt the following 1-form ansatz

$$F = \mathcal{F}^s, \quad (2.15)$$

where

$$\mathcal{F}^s = d\Phi^s \wedge \tau(I_s), \quad s = e, \quad (2.16)$$

$$\mathcal{F}^s = e^{-2\lambda(\varphi)} * (d\Phi^s \wedge \tau(I)), \quad s = m \quad (2.17)$$

and $I_s \in \Omega$. In (2.17) $*$ is $*[g]$ is the Hodge operator on (M, g) . The indices e and m correspond to electric and magnetic p -branes, respectively.

For the potentials in (2.16), (2.17) and the dilatonic scalar fields we put

$$\Phi^s = \Phi^s(u), \quad \varphi^\alpha = \varphi^\alpha(u), \quad (2.18)$$

$s = e, m$; $\alpha = 1, \dots, l$.

From (2.16) and (2.17) we obtain the relations between the dimensions of the p -brane worldsheets and the ranks of forms:

$$d(I_s) = m - 1, \quad d(I_s) = D - m - 1, \quad (2.19)$$

for $s = e, m$, respectively.

It follows from [5] that the equations of motion (2.2)–(2.4) and the Bianchi identities

$$dF = 0, \quad (2.20)$$

for the field configuration (2.7), (2.15)–(2.18) are equivalent to equations of motion for a σ -model with the action

$$S_\sigma = \frac{\theta}{2} \int du \mathcal{N} \left\{ G_{ij} \dot{\phi}^i \dot{\phi}^j + h_{\alpha\beta} \dot{\varphi}^\alpha \dot{\varphi}^\beta + \varepsilon_s \exp[-2U^s(\phi, \varphi)] (\dot{\Phi}^s)^2 - 2\mathcal{N}^{-2} V(\phi) \right\} \quad (2.21)$$

where $\dot{x} \equiv dx/du$,

$$V = V(\phi) = \frac{1}{2} \sum_{i=0}^n \xi_i d_i e^{-2\phi^i + 2\gamma_0(\phi)} \quad (2.22)$$

is the potential with

$$\gamma_0(\phi) = \sum_{i=0}^n d_i \phi^i; \quad (2.23)$$

furthermore,

$$\mathcal{N} = \exp(\gamma_0 - \gamma) > 0 \quad (2.24)$$

is the lapse function,

$$U^s = U^s(\phi, \varphi) = -\chi_s \lambda(\varphi) + \sum_{i \in I_s} d_i \phi^i, \quad (2.25)$$

$$\varepsilon_s = (-\varepsilon[g])^{(1-\chi_s)/2} \varepsilon(I_s) \theta = \pm 1 \quad (2.26)$$

for $s = e, m$, $\varepsilon[g] = \text{sign det}(g_{MN})$, $\chi_s = +1, -1$, for $s = e, m$, respectively, and

$$G_{ij} = d_i \delta_{ij} - d_i d_j \quad (2.27)$$

are components of the “cosmological” minisupermetric, $i, j = 0, \dots, n$ [11].

In the electric case for finite “internal space” volumes V_i the action (2.21) (the time manifold should be S^1) coincides with the action (2.1) if $\theta = -1/\kappa_0^2$, $\kappa^2 = \kappa_0^2 V_0 \dots V_n$.

The action (2.21) may be also written in the form

$$S_\sigma = \frac{\theta}{2} \int du \mathcal{N} \left\{ \mathcal{G}_{\hat{A}\hat{B}}(X) \dot{X}^{\hat{A}} \dot{X}^{\hat{B}} - 2\mathcal{N}^{-2}V \right\} \quad (2.28)$$

where $X = (X^{\hat{A}}) = (\phi^i, \varphi^\alpha, \Phi^s) \in \mathbb{R}^N$, and the minisupermetric $\mathcal{G} = \mathcal{G}_{\hat{A}\hat{B}}(X) dX^{\hat{A}} \otimes dX^{\hat{B}}$ on minisuperspace $\mathcal{M} = \mathbb{R}^N$, $N = n + 2 + l$, is defined by the relation

$$(\mathcal{G}_{\hat{A}\hat{B}}(X)) = \begin{pmatrix} G_{ij} & 0 & 0 \\ 0 & h_{\alpha\beta} & 0 \\ 0 & 0 & \varepsilon_s e^{-2U^s(X)} \end{pmatrix}. \quad (2.29)$$

The minisuperspace metric may be written as follows:

$$\mathcal{G} = \bar{G} + \varepsilon_s e^{-2U^s(x)} d\Phi^s \otimes d\Phi^s \quad (2.30)$$

where $x = (x^A) = (\phi^i, \varphi^\alpha)$, $\bar{G} = \bar{G}_{AB} dx^A \otimes dx^B$,

$$(\bar{G}_{AB}) = \begin{pmatrix} G_{ij} & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix}, \quad (2.31)$$

$U^s(x) = U_A^s x^A$ is defined in (2.25) and

$$(U_A^s) = (d_i \delta_{iI_s}, -\chi_s \lambda_\alpha). \quad (2.32)$$

Here δ_{iI} is an indicator of i belonging to I : $\delta_{iI} = 1$, $i \in I$ and $\delta_{iI} = 0$, $i \notin I$.

The potential (2.22) reads

$$V = \sum_{j=0}^n \frac{\xi_j}{2} d_j e^{2U^j(x)}, \quad (2.33)$$

where

$$U^j(x) = U_A^j x^A = -\phi^j + \gamma_0(\phi), \quad (2.34)$$

$$(U_A^j) = (-\delta_i^j + d_i, 0). \quad (2.35)$$

The integrability of the Lagrange system (2.28) depends on the scalar products of co-vectors U^j , U^s corresponding to \bar{G} :

$$(U, U') = \bar{G}^{AB} U_A U'_B, \quad (2.36)$$

where

$$(\bar{G}^{AB}) = \begin{pmatrix} G^{ij} & 0 \\ 0 & h^{\alpha\beta} \end{pmatrix} \quad (2.37)$$

is the matrix inverse to (2.31). Here (as in [11])

$$G^{ij} = \frac{\delta^{ij}}{d_i} + \frac{1}{2-D}, \quad (2.38)$$

$i, j = 0, \dots, n$. These products have the following form:

$$(U^i, U^j) = \frac{\delta_{ij}}{d_j} - 1, \quad (2.39)$$

$$(U^s, U^s) = d(I_s) + \frac{(d(I_s))^2}{2-D} + \lambda_\alpha \lambda_\beta h^{\alpha\beta}, \quad (2.40)$$

$$(U^s, U^i) = -\delta_{iI_s}, \quad (2.41)$$

$s = e, m$.

2.1. Wheeler–DeWitt equation

Here we fix the gauge as follows:

$$\gamma_0 - \gamma = f(X), \quad \mathcal{N} = e^f, \quad (2.42)$$

where $f: \mathcal{M} \rightarrow \mathbb{R}$ is a smooth function. Then we obtain the Lagrange system with the Lagrangian

$$L_f = \frac{\theta}{2} e^f \mathcal{G}_{\hat{A}\hat{B}}(X) \dot{X}^{\hat{A}} \dot{X}^{\hat{B}} - \theta e^{-f} V \quad (2.43)$$

and the energy constraint

$$E_f = \frac{\theta}{2} e^f \mathcal{G}_{\hat{A}\hat{B}}(X) \dot{X}^{\hat{A}} \dot{X}^{\hat{B}} + \theta e^{-f} V = 0. \quad (2.44)$$

The standard prescriptions of covariant and conformally covariant quantization (see, e.g., [11, 9, 10]) lead to the Wheeler–DeWitt (WDW) equation

$$\hat{H}^f \Psi^f \equiv \left(-\frac{1}{2\theta} \Delta [e^f \mathcal{G}] + \frac{a}{\theta} R [e^f \mathcal{G}] + e^{-f} \theta V \right) \Psi^f = 0 \quad (2.45)$$

where

$$a = a_c(N) = \frac{N-2}{8(N-1)}. \quad (2.46)$$

Here $\Psi^f = \Psi^f(X)$ is the wave function corresponding to the f -gauge (2.42) and satisfying the relation

$$\Psi^f = e^{bf} \Psi^{f=0}, \quad b = (2-N)/2, \quad (2.47)$$

$\Delta[\mathcal{G}_1]$ and $R[\mathcal{G}_1]$ denote the Laplace–Beltrami operator and the scalar curvature corresponding to \mathcal{G}_1 . We note that parameter a may be arbitrary if we do not care about the conformal covariance of the WDW equation.

For the scalar curvature of minisupermetric (2.30) we get (see (2.29) in [5]):

$$R[\mathcal{G}] = -2(U^s, U^s). \quad (2.48)$$

For the Laplace operator we obtain

$$\Delta[\mathcal{G}] = e^{U^s(x)} \frac{\partial}{\partial x^A} \left(\bar{G}^{AB} e^{-U^s(x)} \frac{\partial}{\partial x^B} \right) + \varepsilon_s e^{2U^s(x)} \left(\frac{\partial}{\partial \Phi^s} \right)^2. \quad (2.49)$$

The WDW equation (2.45) for $f = 0$ (in the harmonic time gauge)

$$\hat{H} \Psi = \left(-\frac{1}{2\theta} \Delta[\mathcal{G}] + \frac{a}{\theta} R[\mathcal{G}] + \theta V \right) \Psi = 0, \quad (2.50)$$

may be rewritten, using Eqs. (2.48), (2.49) and

$$U^{si} = G^{ij} U_j^s = \delta_{iI_s} - \frac{d(I_s)}{D-2}, \quad U^{s\alpha} = -\chi_s \lambda^\alpha, \quad (2.51)$$

as follows:

$$\begin{aligned} 2\theta\hat{H}\Psi = & \left\{ -G^{ij}\frac{\partial}{\partial\phi^i}\frac{\partial}{\partial\phi^j} - h^{\alpha\beta}\frac{\partial}{\partial\varphi^\alpha}\frac{\partial}{\partial\varphi^\beta} \right. \\ & \left. - \varepsilon_s e^{2U^s(\phi,\varphi)} \left(\frac{\partial}{\partial\Phi^s} \right)^2 \right. \\ & + \left[\sum_{i \in I_s} \frac{\partial}{\partial\phi^i} - \frac{d(I_s)}{D-2} \sum_{j=0}^n \frac{\partial}{\partial\phi^j} - \chi_s \lambda_{a_s}^\alpha \frac{\partial}{\partial\varphi^\alpha} \right] \\ & \left. + 2aR[\mathcal{G}] + 2\theta^2 V \right\} \Psi = 0. \end{aligned} \quad (2.52)$$

Here $\hat{H} \equiv \hat{H}^{f=0}$ and $\Psi \equiv \Psi^{f=0}$.

3. Exact solutions

Here we use the following restriction on the parameters of the model:

$$(i) \quad \xi_0 = d_0 - 1, \quad \xi_1 = \dots = \xi_n = 0, \quad (3.1)$$

one space $M_0 = S^{d_0}$ is a unit sphere and all M_i ($i > 1$) are Ricci-flat;

$$(ii) \quad 0 \notin I_s, \quad (3.2)$$

i.e. the “brane” submanifold M_{I_s} (see (2.13)) does not contain M_0 , and

$$(iii) \quad (U^s, U^s) > 0. \quad (3.3)$$

The latter is satisfied in most of examples of interest.

From (i), (ii) we get for the potential (2.33):

$$V = \frac{1}{2} \xi_0 d_0 e^{2U^0(x)}, \quad (3.4)$$

where (see (2.39))

$$(U^0, U^0) = 1/d_0 - 1 < 0. \quad (3.5)$$

From (iii) and (2.41) we get

$$(U^0, U^s) = 0. \quad (3.6)$$

3.1. Classical solutions

Consider a solution to the Lagrange equations corresponding to the Lagrangian (2.43) with the energy-constraint (2.44) under the restrictions (3.1)–(3.3). We put $f = 0$, i.e., use the harmonic time gauge.

Integrating the Maxwell equations (for $s = e$) and the Bianchi identities (for $s = m$), we get

$$\frac{d}{du} \left(\exp(-2U^s) \dot{\Phi}^s \right) = 0 \iff \dot{\Phi}^s = Q_s \exp(2U^s), \quad (3.7)$$

where $Q_s = \text{const}$. We put $Q_s \neq 0$.

For fixed Q_s , the Lagrange equations for the Lagrangian (2.43) with $f = 0$ corresponding to $(x^A) =$

(ϕ^i, φ^α) , with Eqs. (3.7) substituted, are equivalent to the Lagrange equations for the Lagrangian

$$L_Q = \frac{1}{2} \bar{G}_{AB} \dot{x}^A \dot{x}^B - V_Q \quad (3.8)$$

where

$$V_Q = V + \frac{1}{2} \varepsilon_s Q_s^2 \exp[2U^s(x)], \quad (3.9)$$

(\bar{G}_{AB}) and V are defined in (2.31) and (2.22), respectively. The zero-energy constraint (2.44) reads

$$E_Q = \frac{1}{2} \bar{G}_{AB} \dot{x}^A \dot{x}^B + V_Q = 0. \quad (3.10)$$

When the conditions (i)–(iii) are satisfied, exact solutions for the Lagrangian (3.8) with the potential (3.9) and V from (3.4) have the following form [14]:

$$\begin{aligned} x^A(u) = & -\frac{U^{0A}}{(U^0, U^0)} \ln |f_0(u - u_0)| \\ & - \frac{U^{sA}}{(U^s, U^s)} \ln |f_s(u - u_s)| + c^A u + \bar{c}^A \end{aligned} \quad (3.11)$$

where u_0, u_s are constants,

$$f_0(\tau) = \frac{(d_0 - 1)}{\sqrt{C_0}} \sinh(\sqrt{C_0}\tau), \quad (3.12)$$

and

$$f_s(\tau) = \frac{|Q_s|}{\nu_s \sqrt{C_s}} \sinh(\sqrt{C_s}\tau), \quad \varepsilon_s < 0; \quad (3.13)$$

$$\frac{|Q_s|}{\nu_s \sqrt{C_s}} \cosh(\sqrt{C_s}\tau), \quad C_s > 0, \quad \varepsilon_s > 0 \quad (3.14)$$

C_0 and C_s are constants. Here $\sinh(\sqrt{C}x)/\sqrt{C} = x$ for $C = 0$.

The contravariant components $U^{0A} = \bar{G}^{AB} U_B^0$ are

$$U^{0i} = -\frac{\delta_0^i}{d_0}, \quad U^{0\alpha} = 0. \quad (3.15)$$

(For U^{sA} , $s \in S$, see (2.51)).

The vectors $c = (c^A)$ and $\bar{c} = (\bar{c}^A)$ satisfy the linear constraint relations (due the configuration space splitting into a sum of three mutually orthogonal subspaces, see [14])

$$U^0(c) = c^0 + \sum_{j=0}^n d_j c^j = 0, \quad U^0(\bar{c}) = 0, \quad (3.16)$$

$$U^s(c) = \sum_{i \in I_s} d_i c^i - \chi_s \lambda_{a_s \alpha} c^\alpha = 0, \quad U^s(\bar{c}) = 0. \quad (3.17)$$

The zero-energy constraint $E = E_0 + E_s + (1/2) \times \bar{G}_{AB} c^A c^B = 0$, with $C_0 = 2E_0(U^0, U^0)$, $C_s = 2E_s(U^s, U^s)$ may be written as

$$\begin{aligned} C_0 \frac{d_0}{d_0 - 1} = & C_s \nu_s^2 + h_{\alpha\beta} c^\alpha c^\beta + \sum_{i=1}^n d_i (c^i)^2 \\ & + \frac{1}{d_0 - 1} \left(\sum_{i=1}^n d_i c^i \right)^2. \end{aligned} \quad (3.18)$$

The following expressions for the metric and scalar fields follows from (2.51 (3.15)) and (3.15):

$$g = [f_s^2(u - u_s)]^{d(I_s)\nu_s^2/(D-2)} \times \left\{ [f_0^2(u - u_0)]^{d_0/(1-d_0)} e^{2c^0 u + 2\bar{c}^0} [du \otimes du + f_0^2(u - u_0)g^0] + \sum_{i \neq 0} [f_s^2(u - u_s)]^{-\nu_s^2 \delta_{iI_s}} e^{2c^i u + 2\bar{c}^i} g^i \right\}, \quad (3.19)$$

$$\varphi^\alpha = \nu_s^2 \chi_s \lambda_{a_s}^\alpha \ln |f_s| + c^\alpha u + \bar{c}^\alpha. \quad (3.20)$$

From the relation $\exp(2U^s) = f_s^{-2}$ (following from (3.6), (3.11) (3.17)) we get for the forms:

$$\mathcal{F}^s = Q_s f_s^{-2} du \wedge \tau(I_s), \quad (3.21)$$

$$\bar{\mathcal{F}}^s = e^{-2\lambda(\varphi)} * [Q_s f_s^{-2} du \wedge \tau(I_s)] = \bar{Q}_s \tau(\bar{I}_s) \quad (3.22)$$

for $s = e, m$, respectively, where $\bar{Q}_s = Q_s \varepsilon(I_s) \mu(I_s)$ and $\mu(I) = \pm 1$ is defined by the relation $\mu(I) du \wedge \tau(I_0) = \tau(\bar{I}) \wedge du \wedge \tau(I)$ [14].

Thus we obtain exact spherically symmetric solutions with internal Ricci-flat spaces (M_i, g^i) , $i = 2, \dots, n$ in the presence of several scalar fields and one form. The solution is presented by the relations (3.20), (3.21)–(3.19) with the functions f_0 , f_s defined in (3.12)–(3.14) and the relations (3.16)–(3.17), (3.18) on the solution parameters c^A , \bar{c}^A ($A = i, \alpha$), C_0 , C_s , ν_s .

This solution describes a charged p -brane (electric or magnetic) “living” on the submanifold M_{I_s} (2.13), where the set I_s does not contain 0, i.e. the p -brane lives only in the “internal” Ricci-flat spaces.

In the non-composite case with several intersecting p -branes, solutions of this type were considered in [12, 13] (the electric case) and [15] (the electro-magnetic case). For the composite case see [14].

3.2. Quantum solutions

The truncated minisuperspace metric (2.31) may be diagonalized by the linear transformation

$$z^A = S^A_B x^B, \quad (z^A) = (z^0, z^a, z^s) \quad (3.23)$$

as follows:

$$\bar{G} = -dz^0 \otimes dz^0 + \eta_s dz^s \otimes dz^s + dz^a \otimes dz^b \eta_{ab}, \quad (3.24)$$

where $a, b = 1, \dots, n$; $\eta_{ab} = \eta_{aa} \delta_{ab}$; $\eta_{aa} = \pm 1$, and

$$q_0 z^0 = U^0(x), \quad q_s z^s = U^s(x), \quad (3.25)$$

with $q_0 = |(U^0, U^0)|^{1/2} = [1 - 1/d_0]^{1/2} > 0$, $q_s = \nu_s^{-1} = |(U^s, U^s)|^{1/2}$.

From (2.49), (3.23), (3.24) we get

$$\Delta[\mathcal{G}] = - \left(\frac{\partial}{\partial z^0} \right)^2 + \eta^{ab} \frac{\partial}{\partial z^a} \frac{\partial}{\partial z^b} + e^{q_s z^s} \frac{\partial}{\partial z^s} \left(e^{-q_s z^s} \frac{\partial}{\partial z^s} \right) + \varepsilon_s e^{2q_s z^s} \left(\frac{\partial}{\partial \Phi^s} \right)^2. \quad (3.26)$$

As usual, we seek a solution to the WDW equation (2.50) by separation of variables, i.e., we put

$$\Psi_*(z) = \Psi_0(z^0) \Psi_s(z^s) e^{iP_s \Phi^s} e^{ip_a z^a}. \quad (3.27)$$

It follows from (3.26) that $\Psi_*(z)$ satisfies the WDW equation (2.50) if

$$2\hat{H}_0 \Psi_0 \equiv \left\{ \left(\frac{\partial}{\partial z^0} \right)^2 + \theta^2 \xi_0 d_0 e^{2q_0 z^0} \right\} \Psi_0 = 2\mathcal{E}_0 \Psi_0; \quad (3.28)$$

$$2\hat{H}_s \Psi_s \equiv \left\{ -e^{q_s z^s} \frac{\partial}{\partial z^s} \left(e^{-q_s z^s} \frac{\partial}{\partial z^s} \right) + \varepsilon_s P_s^2 e^{2q_s z^s} \right\} \Psi_s = 2\mathcal{E}_s \Psi_s, \quad (3.29)$$

and

$$2\mathcal{E}_0 + \eta^{ab} p_a p_b + 2\mathcal{E}_s + 2aR[\mathcal{G}] = 0, \quad (3.30)$$

with a and $R[\mathcal{G}]$ from (2.46) and (2.48), respectively.

Linearly independent solutions to Eqs.(3.28) and (3.29) have the following form:

$$\Psi_0(z^0) = B_{\omega_0}^0 \left(\sqrt{-\theta^2(d_0 - 1)d_0} \frac{e^{q_0 z^0}}{q_0} \right), \quad (3.31)$$

$$\Psi_s(z^s) = e^{q_s z^s/2} B_{\omega_s}^s \left(\sqrt{\varepsilon_s P_s^2} \frac{e^{q_s z^s}}{q_s} \right), \quad (3.32)$$

where

$$\omega_0 = \sqrt{2\mathcal{E}_0}/q_0, \quad \omega_s = \sqrt{\frac{1}{4} - 2\mathcal{E}_s \nu_s^2}, \quad (3.33)$$

$B_\omega^0, B_\omega^s = I_\omega, K_\omega$ are the modified Bessel function.

The general solution of the WDW equation (2.50) is a superposition of the “separated” solutions (3.27):

$$\Psi(z) = \sum_B \int dp dP d\mathcal{E} C(p, P, \mathcal{E}, B) \Psi_*(z|p, P, \mathcal{E}, B), \quad (3.34)$$

where $p = (p_a)$, $P = (P_s)$, $\mathcal{E} = (\mathcal{E}_s)$, $B = (B^0, B^s)$, $B^0, B^s = I, K$, and $\Psi_* = \Psi_*(z|p, P, \mathcal{E}, B)$ is given by the relations (3.27), (3.31)–(3.33) with \mathcal{E}_0 from (3.30). Here $C(p, P, \mathcal{E}, B)$ are smooth enough functions. For several intersecting p -branes (non-composite electric and composite electro-magnetic) see [12] and [14], respectively.

4. Black hole (BH) solutions

4.1. Classical BH solutions

Let us single out solutions with a horizon (with respect to time t). We put

$$1 \in I_s, \quad (4.1)$$

i.e., the p -brane contains the time manifold. Let

$$\varepsilon_s = -1 \quad (4.2)$$

This is a physical restriction satisfied when a pseudo-Euclidean brane in pseudo-Euclidean space is considered.

We single out the solution with a horizon: for integration constants we put $\bar{c}^A = 0$,

$$c^A = \bar{\mu} \sum_{r=0,s} \frac{U^{rA}}{(U^r, U^r)} - \bar{\mu} \delta_1^A, \quad (4.3)$$

$$C_0 = C_s = \bar{\mu}^2, \quad (4.4)$$

where $\bar{\mu} > 0$. Here $A = (i_A, \alpha_A)$ and $A = 1$ means $i_A = 1$. It may be verified that the restrictions (3.16)–(3.17) and (3.18) are satisfied identically.

Let us introduce the new radial variable $R = R(u)$ by the relations

$$e^{-2\bar{\mu}u} = 1 - \frac{2\mu}{R^{\bar{d}}}, \quad \mu = \bar{\mu}\bar{d} > 0, \quad \bar{d} = d_0 - 1 \quad (4.5)$$

and put $u_0 = 0$, $u_s < 0$,

$$\frac{|Q_s|}{\bar{\mu}\nu_s} \sinh \beta_s = 1, \quad \beta_s \equiv \bar{\mu}|u_s|, \quad (4.6)$$

$s = e, m$.

Then the solutions for the metric and the scalar fields (see (3.20), (3.19)) are:

$$g = H_s^{2d(I_s)\nu_s^2/(D-2)} \left\{ \frac{dR \otimes dR}{1 - 2\mu/R^{\bar{d}}} + R^2 d\Omega_{d_0}^2 - H_s^{-2\nu_s^2} \left(1 - \frac{2\mu}{R^{\bar{d}}} \right) dt \otimes dt + \sum_{i=2}^n H_s^{-2\nu_s^2 \delta_{iI_s}} g^i \right\}, \quad (4.7)$$

$$\varphi^\alpha = \nu_s^2 \chi_s \lambda_{\alpha s}^\alpha \ln H_s, \quad (4.8)$$

where

$$H_s = 1 + \frac{\mathcal{P}_s}{R^{\bar{d}}}, \quad \mathcal{P}_s \equiv \frac{|Q_s|\bar{d}}{\nu_s} e^{-\beta_s}. \quad (4.9)$$

The form field is given by (2.16), (2.17) with

$$\Phi^s = \frac{\nu_s}{H_s'}, \quad (4.10)$$

$$H_s' = 1 + \frac{\mathcal{P}_s'}{R^{\bar{d}} + \mathcal{P}_s - \mathcal{P}_s'}, \quad (4.11)$$

$$\mathcal{P}_s' \equiv -\frac{Q_s \bar{d}}{\nu_s}. \quad (4.12)$$

$s = e, m$. It follows from (4.6), (4.9) and (4.12) that

$$|\mathcal{P}_s'| = \frac{\mu}{\sinh \beta_s} = \mathcal{P}_s e^{\beta_s} = \sqrt{\mathcal{P}_s(\mathcal{P}_s + 2\mu)}. \quad (4.13)$$

The Hawking “temperature” corresponding to the solution is (see also [15, 18])

$$T_H = \frac{\bar{d}}{4\pi(2\mu)^{1/\bar{d}}} \left(\frac{2\mu}{2\mu + \mathcal{P}_s} \right)^{\nu_s^2}. \quad (4.14)$$

Recall that $\nu_s^2 = (U^s, U^s)^{-1}$.

Extremal case. In the extremal case $\mu \rightarrow +0$ we get for the metric (4.7)

$$g = H_s^{2d(I_s)\nu_s^2/(D-2)} \left\{ dR \otimes dR + R^2 d\Omega_{d_0}^2 - H_s^{-2\nu_s^2} dt \otimes dt + \sum_{i=2}^n H_s^{-2\nu_s^2 \delta_{iI_s}} g^i \right\}, \quad (4.15)$$

where $H_s = H_s'$ ($\mathcal{P}_s' = \mathcal{P}_s$) in (4.10), $s = e, m$.

Remark. This solution has a regular horizon at $R \rightarrow +0$ (with a finite limit of the Riemann tensor squared for $R \rightarrow +0$) if $\nu_s^2 d(I_s) \bar{d} \geq D - 2$ (see [19]).

4.2. Quantum analogues of BH solutions

Let us put $a = 0$ in the WDW equation. In this case there exist a map that puts into correspondence some quantum solution to WDW equation to any classical BH solution. We also put

$$\mathcal{E}_0 = E_0, \quad \mathcal{E}_s = E_s, \quad (4.16)$$

i.e., the classical energies of subsystems coincide with the eigenvalues of the Hamiltonians \hat{H}_0 and \hat{H}_s , respectively, and

$$P_s = Q_s = -\mathcal{P}_s' \frac{\nu_s}{\bar{d}}, \quad (4.17)$$

i.e., the classical charges coincide with the eigenvalues of the momentum operators $\hat{P}_s = -\partial/\partial\Phi^s$.

Then it may be shown that

$$p_a z^a = c_A x^A. \quad (4.18)$$

Using these relations, we get solutions to the WDW equation (with $a = 0$) of special type. These solutions correspond to classical BH solutions and have an ambiguity in the choice of the Bessel functions. We note that the quantum energy constraint (3.30) is satisfied identically due to our choice $a = 0$.

Extremal case. In the extremal case we have

$$\mathcal{E}_0 = \mathcal{E}_s = p_a z^a = 0. \quad (4.19)$$

Hence the wave function (4.20) reads

$$\Psi_* = \Psi_0 \Psi_s \exp(iQ_s), \quad (4.20)$$

where

$$\Psi_0 = B_0^0(i|\theta|r_0 v^0), \quad (4.21)$$

$$\Psi_s = v_s^{1/2} B_{1/2}^s(iQ_s v^s), \quad (4.22)$$

$r_0 = \sqrt{d_0(d_0 - 1)}$ and

$$v_0 = e^{q_0 z^0} = \exp(-\phi^0 + \sum_{i=0}^n d_i \phi^i), \quad (4.23)$$

$$v_s = e^{q_s z^s} = \exp(-\chi_s \lambda(\varphi) + \sum_{i \in I_s} d_i \phi^i), \quad (4.24)$$

are “quasivolumes”, $s = e, m$. The gravitational part of the wave function, i.e. Ψ_0 , coincides with that of Refs. [22, 23], see also [21, 24]. For small quasivolumes $v_s \rightarrow 0$ we get

$$\Psi_s \sim v^s \sqrt{2iQ_s/\pi}, \quad B = I, \quad (4.25)$$

$$\Psi_s \sim \sqrt{\pi/2iQ_s}, \quad B = K. \quad (4.26)$$

For a big brane quasivolume $v_s \rightarrow \infty$ we get

$$\Psi_s \sim \frac{\exp(iQ_s v^s)}{\sqrt{2\pi i Q_s}}, \quad B = I, \quad (4.27)$$

$$\Psi_s \sim \frac{\exp(-iQ_s v^s)}{\sqrt{2iQ_s/\pi}}, \quad B = K, \quad (4.28)$$

Thus, for a positive charge \mathcal{P}_s or, equivalently, $Q_s < 0$ (see (4.17)), the brane part of the solution with $B = K$ satisfies the outgoing-wave boundary condition, used first in quantum cosmology in [25], and is regular for small brane quasi-volumes.

4.2.1. Example: $D = 11$ supergravity

Consider $D = 11$ supergravity [7] with the truncated bosonic action (without a Chern-Simons term)

$$S_{11tr} = \int_M d^{11}z \sqrt{|g|} \left\{ R[g] - \frac{1}{4!} F^2 \right\}, \quad (4.29)$$

where F is a 4-form. Here we have two types of solutions: an electric 2-brane with $d(I_s) = 3$ ($s = e$) and a magnetic 5-brane with $d(I_s) = 6$ ($s = m$). In both cases $(U^s, U^s) = 2 = \nu_s^{-2}$, ($s = e, m$). We put $\varepsilon(1) = -1$ and $\varepsilon(k) = 1$, $k > 1$.

In the extremal case we get for an $M2$ -brane ($s = e$) and an $M5$ -brane ($s = m$):

$$\begin{aligned} \Psi_* &= B_0^0 (i|\theta| r_0 v^0) v_s^{1/2} \times \\ &\times B_{1/2}^s \left(-i\mathcal{P}_s \frac{1}{2d} v^s \right) \exp \left(-i\mathcal{P}_s \frac{\nu_s}{d} \Phi^s \right), \end{aligned} \quad (4.30)$$

with $v_s = \exp(\sum_{i \in I_s} d_i \phi^i)$, $s = e, m$, and v_0 defined in (4.23).

4.2.2. Example: $D = 4$ Einstein-Maxwell gravity

Consider $D = 4$ gravity:

$$S_4 = \int_M d^4z \sqrt{|g|} \left\{ R[g] - \frac{1}{2!} F^2 \right\}, \quad (4.31)$$

where F is a 2-form. Here we have two types of solutions: an electric 0-brane (electric charge) with $d(I_s) = 1$ ($s = e$) and a magnetic 0-brane (magnetic charge) with $d(I_s) = 1$ ($s = m$). In both cases $(U^s, U^s) = 1/2 = \nu_s^{-2}$ ($s = e, m$). We put $\varepsilon(1) = -1$. Here $n = 1$, $d_0 = 2$ and $d_1 = 1$.

The (extremal) solutions read

$$\begin{aligned} \Psi_* &= B_0^0 (i|\theta| \sqrt{2} v^0) v_s^{1/2} \times \\ &\times B_{1/2}^s (-i\mathcal{P}_s v^s) \exp(-i\mathcal{P}_s \nu_s \Phi^s), \end{aligned} \quad (4.32)$$

with $v_s = \exp(\phi^1)$, $s = e, m$, and $v_0 = \exp(\phi^0 + \phi^1)$. We see that here electric and magnetic solutions coincide. (This fact may be considered as a simple manifestation of electro-magnetic duality at a quantum level).

4.2.3. WDW equation with fixed charges

There exists another quantization scheme, where the fields of forms are considered to be classical. This scheme is based on the zero-energy constraint relation (3.10), see [20]. The corresponding WDW equation in the harmonic gauge reads

$$\hat{H}_Q \Psi \equiv \left(-\frac{1}{2\theta} \bar{G}^{AB} \frac{\partial}{\partial x^A} \frac{\partial}{\partial x^B} + \theta V_Q \right) \Psi = 0 \quad (4.33)$$

where the potential V_Q is defined in (3.9). This equation describes quantum cosmology with classical fields of forms and quantum scale factors and dilatonic fields.

The basis of solutions is given by the following replacements in (3.27), (3.30), (3.32) and (3.33): $P_s \mapsto Q_s$, $2aR[\mathcal{G}] \mapsto 0$, $\omega_s \mapsto \sqrt{-2\mathcal{E}_s \nu_s^2}$, and $\Psi_s(z^s) \mapsto B_{\omega_s}^s \left(\sqrt{\varepsilon_s Q_s^2} e^{q_s z^s} / q_s \right)$.

In this approach there is no problem with the a parameter when quantum analogues of black-hole solutions are constructed (since in the harmonic gauge $R(\bar{G}) = 0$).

Let us compare quantum black-hole solutions in the two approaches. The function Ψ_0 is the same in both approaches but for the brane part of the wave function we get

$$\Psi_s = B_0^s (iQ_s v^s). \quad (4.34)$$

For small quasivolumes $v_s \rightarrow 0$ we get

$$\Psi_s \sim 1, \quad B = I, \quad (4.35)$$

$$\Psi_s \sim -\ln(iQ_s v^s/2), \quad B = K. \quad (4.36)$$

For a big brane quasivolume, $v_s \rightarrow \infty$, we get

$$\Psi_s \sim \frac{\exp(iQ_s v^s)}{\sqrt{2\pi i Q_s v^s}}, \quad B = I, \quad (4.37)$$

$$\Psi_s \sim \frac{\exp(-iQ_s v^s)}{\sqrt{2iQ_s v^s/\pi}}, \quad B = K, \quad (4.38)$$

Thus for $Q_s < 0$ the brane part of the solution with $B = K$ satisfies the outgoing-wave boundary condition [25] but it is not regular for a small brane quasi-volume.

5. Conclusions

We have considered classical spherically symmetric solutions with one brane and the corresponding black-hole solutions. Using solutions to the Wheeler-DeWitt equation, have we suggested the quantum analogues to black-hole solutions in the extremal and non-extremal cases. This was possible when the coupling parameter in the WDW equation was trivial: $a = 0$. In the alternative approach of [20] (with classical fields of forms) one may use an arbitrary coupling a in the WDW equation when constructing quantum analogues.

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